Dirac delta function

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Version 1.0

1 Heuristic introduction

1.1 Preliminary example

We have seen in the previous lecture that the Heaviside step function is used to model the classical behavior of an electrical circuit. In mechanics, an impulse is usually applied on a mechanical system, to provide momentum for example.

> **Example 1.1**
> Consider a Galilean reference and, within it, a point mobile object of mass m, lying on a flat and horizontal support.
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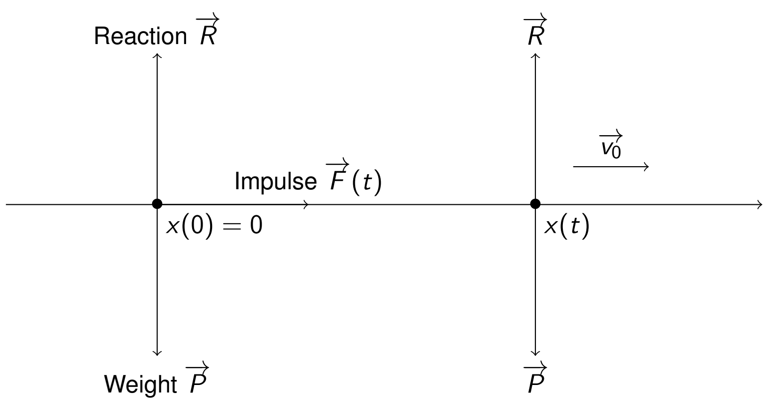
> 
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>
> We neglect any spurious force (friction, rolling, sliding, etc.). To simplify our study, we assume that the object can only move in one direction, and we denote $x(t)$ its instantaneous position at a given time t relatively to a defined origin. This position is the response of the system. We define $v(t) = x'(t)$ its instantaneous speed and $a(t) = v'(t) = x''(t)$ its instantaneous acceleration. We assume that the object is initially at rest, and weight and reaction, which are the only forces applied to this system, compensate, so that for any $t < 0$, $a(t) = v(t) = 0$ et $x(t) = 0$ is the initial position of the resting object used as origin. At time $t = 0$, we apply an impulse on the object, corresponding to a force $F(t)$ giving the direction of motion of the object, so that it instantaneously acquires speed $v_0 > 0$. For $t > 0$, no additional force is applied, so that the object is in uniform linear motion with speed $v_0 > 0$, i.e. $v(t) = v_0 \Upsilon(t)$. The position, response to the input $F(t)$, is then $x(t) = v_0 t \Upsilon(t)$.
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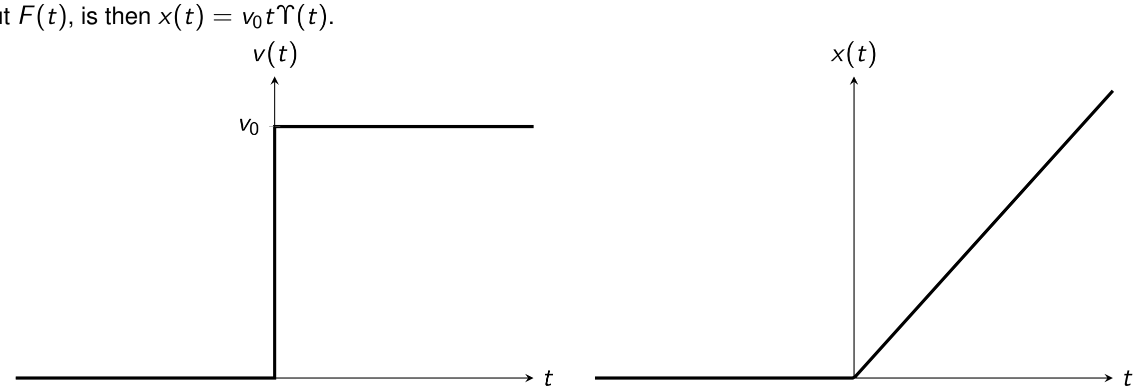
> 
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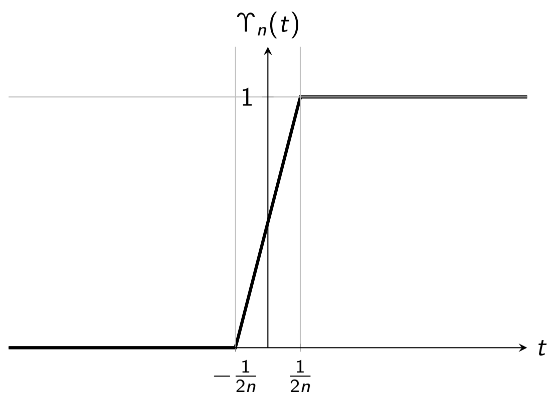
By the second law of motion, we have $ma(t) = mv'(t) = F(t)$, thus $F(t) = mv_0\Upsilon'(t)$. This example shows the necessity to carefully define the derivative of step function $\Upsilon(t)$.

1.2 Approximating the step function with a sequence of functions

We have seen in the RC circuit example that function Υ is differentiable over $\mathbb{R}^*$ and for any $t \in \mathbb{R}^*$, $\Upsilon'(t) = 0$. The main issue is the discontinuity of Υ in 0, making it non-differentiable at this point. First, we try to interpret this derivative by approximating Υ with a sequence of functions $(\Upsilon_n)_{n\in\mathbb{N}^*}$, and studying the behavior of the sequence of derivative functions $(\Upsilon'_n)_{n\in\mathbb{N}^*}$. We define, for any $n \in \mathbb{N}^*$, the piecewise linear function:

$$\Upsilon_n : \mathbb{R} \to \mathbb{R} \qquad t \mapsto \begin{cases} 0 & \text{if } t < -\frac{1}{2n} \\ nt + \frac{1}{2} & \text{if } -\frac{1}{2n} < t < \frac{1}{2n} \\ 1 & \text{if } t > \frac{1}{2n} \end{cases}$$

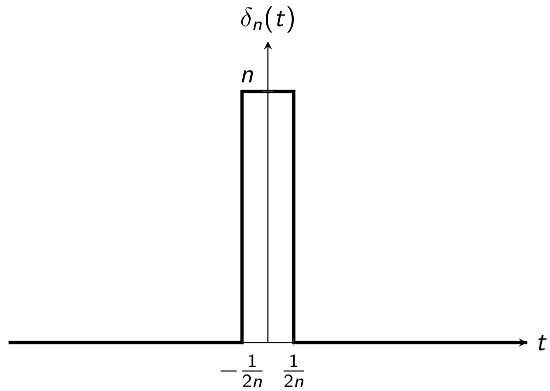


This sequence converges pointwise to Υ, i.e. for almost any $t \in \mathbb{R}$, $\Upsilon(t) = \lim_{n\to+\infty} \Upsilon_n(t)$. Note that for n large enough, function Υ_n is a more realistic model of the state transition from off to on by eliminating the discontinuity in 0 Recall that in order to let a limit function inherits the regularity properties (continuity, differentiability, ...) of the elements of the sequence, the convergence has to be uniform, in other words for $\varepsilon > 0$ arbitrarily small, there exists an index $N \in \mathbb{N}^*$ such that for any $n > N$, the graphical representation of Υ_n is "stuck" between the representations of $\Upsilon - \varepsilon$ and $\Upsilon + \varepsilon$, which can be mathematically translated:

$$\forall \varepsilon > 0 \quad \exists N \in \mathbb{N}^* \quad \forall n > N \quad \forall t \in \mathbb{R} \quad |\Upsilon_n(t) - \Upsilon(t)| < \varepsilon$$

With this definition, there is no uniform convergence in this case since for any $n \in \mathbb{N}^*$, $\Upsilon_n(0) = \frac{1}{2}$, whereas $\lim_{t\to 0^-} \Upsilon(t) = 0$ and $\lim_{t\to 0^+} \Upsilon(t) = 1$. Moreover, functions Υ_n are continuous in 0 whereas Υ has a discontinuity in 0, showing that it does not inherit the continuity property from uniform convergence. Strictly speaking, we cannot consider the limit of sequence $(\Upsilon'_n)_{n\in\mathbb{N}^*}$ as the derivative of Υ. However, the study of this limit will give us an insight into the behavior of the derivative of Υ. For any $n \in \mathbb{N}^*$, we set $\delta_n = \Upsilon'_n$ and define δ as the limit of this sequence, if it exists. For any $n \in \mathbb{N}^*$, function δ_n is piecewise constant and

$$\delta_n(t) = \begin{cases} n & \text{if } -\frac{1}{2n} < t < \frac{1}{2n} \\ 0 & \text{otherwise} \end{cases}$$



Then for any $t \neq 0$, $\delta(t) = 0$, since there exists an index $N \in \mathbb{N}^*$ such that $|t| > \dfrac{1}{2n}$ for any $n > N$. Moreover, $\delta(0) = \lim\limits_{n \to +\infty} n = +\infty$, thus by extending the image set to $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$, we obtain the following definition:

$$\delta : \mathbb{R} \to \overline{\mathbb{R}} \qquad t \mapsto \begin{cases} +\infty & \text{if } t = 0 \\ 0 & \text{otherwise} \end{cases}$$

However, a surprising result appears when we compute the following integral:

$$\forall n \in \mathbb{N}^* \qquad I_n = \int_{-\infty}^{+\infty} \delta_n(t)dt = \int_{-1/2n}^{1/2n} ndt = 1$$

Taking the limit, we get $\displaystyle\int_{-\infty}^{+\infty} \delta(t)dt = 1$. We have a function whose support is restricted to singleton $\{0\}$, the image on this support is $+\infty$ and the integral over $\mathbb{R}$ is 1, going against the rule $0 \times (+\infty) = 0$ established in measure theory. Thereby, we cannot consider δ as a classical measurable or integrable function, but as a more general mathematical object called a distribution.

1.3 Linear forms and distributions

In practice, evaluating the image $f(x_0)$ of function f with real argument $x_0 \in \mathbb{R}$ is generally impossible because it will require the knowledge of the infinite decimal development of both argument and image. It is more appropriate to represent this evaluation by

$$f(x_0) = \lim_{\varepsilon \to 0} \int_{-\infty}^{+\infty} f(x)\varphi_\varepsilon(x - x_0)dx$$

where φ_ε is a function whose support is concentrated around 0. This equality symbolizes the measure of a physical quantity f by a measuring tool φ_ε, all the more precise as ε is small. More generally, for any fixed function $g \in \mathcal{F}(\mathbb{R}, \mathbb{R})$, we notice that application

$$L_g : \mathcal{F}(\mathbb{R}, \mathbb{R}) \to \mathbb{R} \qquad f \mapsto \int_{-\infty}^{+\infty} f(t)g(t)dt$$

is a linear form, also denoted with the duality bracket $L_g(f) = \langle L_g, f \rangle = \langle g, f \rangle$. However, the space of linear forms from $\mathcal{F}(\mathbb{R}, \mathbb{R})$ to $\mathbb{R}$ contains more elements than the linear forms like L_g. We are going to express another linear form, associated with the Dirac delta function.

For any $n \in \mathbb{N}^*$, we define

$$L_n : \mathcal{F}(\mathbb{R},\mathbb{R}) \to \mathbb{R} \qquad f \mapsto \int_{-\infty}^{+\infty} f(t)\delta_n(t)dt$$

and we want to determine the limit L of this sequence. We consider signals f which are continuous in a neighborhood of 0. We fix such a signal. Then for any $n \in \mathbb{N}^*$,

$$L_n(f) - f(0) = \int_{-\infty}^{+\infty} f(t)\delta_n(t)dt - f(0) = n\int_{-1/2n}^{1/2n} (f(t) - f(0))dt$$

Let $\varepsilon > 0$. Using the continuity of f in 0, there exists $\eta > 0$ such that for any $t \in]-\eta, \eta[$, $|f(t) - f(0)| < \varepsilon$. There exists an index $N \in \mathbb{N}^*$ such that for any $n > N$, $\dfrac{1}{2n} < \eta$, which gives

$$|L_n(f) - f(0)| \leq n\int_{-1/2n}^{1/2n} |f(t) - f(0)|dt < \varepsilon$$

We deduce that the sequence of real numbers $(L_n(f))_{n\in\mathbb{N}^*}$ converges to $f(0)$, thus we can define the limit linear form by:

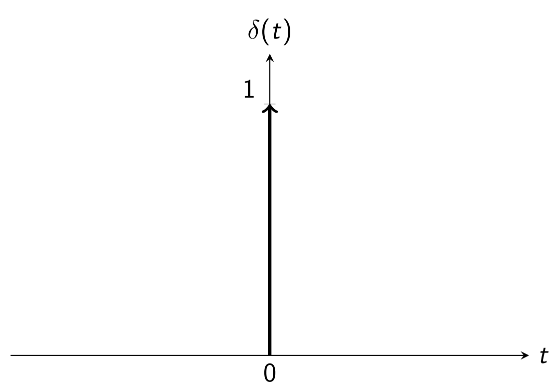
$$L : \mathcal{F}(\mathbb{R},\mathbb{R}) \to \mathbb{R} \qquad f \mapsto f(0)$$

that, by abuse of notation, we write $L : f \mapsto \displaystyle\int_{-\infty}^{+\infty} f(t)\delta(t)dt$, while it would be better to write $L(f) = \langle L, f\rangle = \langle \delta, f\rangle$. In this context, we do not define δ as the limit of the sequence of functions (δ_n) anymore, but as the distribution associated with linear form L, the limit of sequence $(L_n) = (L_{\delta_n})$. Now we have all the ingredients to properly define Dirac [1] delta function. In this context, the word function is an abuse, since δ is a distribution, or sometimes also called a generalized function.

2 Definition of Dirac delta function

Definition 2.1 (Dirac delta function)
Dirac delta function is the distribution $\delta : \mathbb{R} \to \overline{\mathbb{R}}$ satisfying the following properties:

- For any $t \in \mathbb{R}^*$, $\delta(t) = 0$;
- $\delta(0) = +\infty$
- For any signal f defined and continuous in a vicinity of 0, $\displaystyle\int_{-\infty}^{+\infty} f(t)\delta(t)dt = f(0)$.



[1] Paul Dirac (1902-1984), British mathematician and physicist

Remarks:

(i) The property $\int_{-\infty}^{+\infty} \delta(t)dt = 1$ corresponds to the particular case of constant function $f : t \mapsto 1$.

(ii) It is also possible to define the Dirac delta function as the distributional derivative of the Heaviside step function through integration by parts. Indeed, assuming that the derivative Υ' exists, we consider a differentiable signal f et we want to evaluate $I = \int_{-\infty}^{+\infty} f(t)\Upsilon'(t)dt$. Setting, for any $A > 0$, $I(A) = \int_{-A}^{A} f(t)\Upsilon'(t)dt$, integration by parts gives

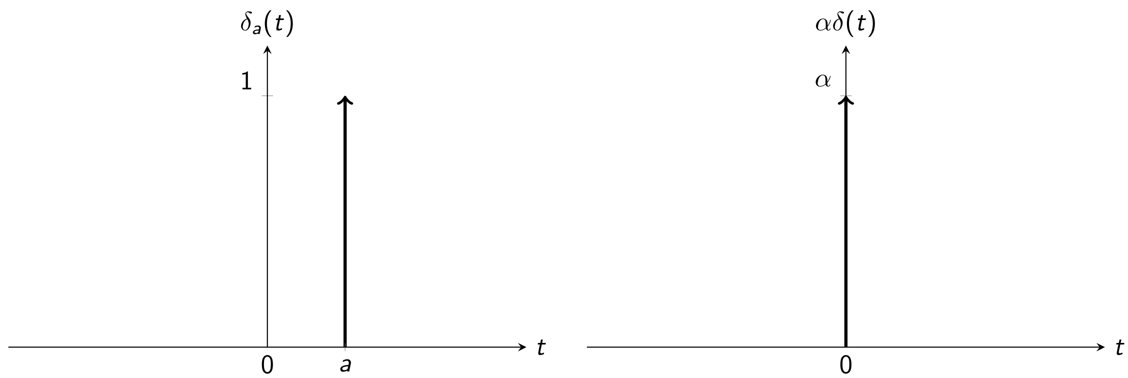
$$I(A) = \int_{-A}^{A} f(t)\Upsilon'(t)dt = \Big[f(t)\Upsilon(t)\Big]_{-A}^{A} - \int_{-A}^{A} f'(t)\Upsilon(t)dt = f(A) - \int_{0}^{A} f'(t)dt = f(0)$$

thus $I = \lim_{A\to+\infty} I(A) = f(0)$.

(iii) So far we have defined the delta function centered in zero, i.e. $\delta_0 = \delta$ corresponding to the linear form $f \mapsto f(0)$. We can also define for any $a \in \mathbb{R}$ the Dirac delta function centered in a by $\delta_a(t) = \delta(t-a)$, corresponding to linear form $f \mapsto f(a)$, i.e.

$$\int_{-\infty}^{+\infty} f(t)\delta_a(t)dt = \int_{-\infty}^{+\infty} f(t)\delta(t-a)dt = \int_{-\infty}^{+\infty} f(t)\delta(a-t)dt = f(a)$$

(iv) By linearity, we can define for any $\alpha \in \mathbb{R}$ the weighted Dirac delta function $\alpha\delta$ as the derivative of the weighted Heaviside step function $\alpha\Upsilon$. It corresponds to linear form $f \mapsto \alpha f(0)$.



(v) We can define a distribution $f\delta$, product of a regular function f by the Dirac delta function δ. Indeed, for any function h,

$$\langle f\delta, h\rangle = \int_{-\infty}^{+\infty} h(t)\left(f(t)\delta(t)\right)dt = \int_{-\infty}^{+\infty} \left(h(t)f(t)\right)\delta(t)dt = h(0)f(0)$$

This product distribution is thus the weighted delta function $f(0)\delta$. This definition has a particular interest when we look for the derivative of a function of the form $g = f\Upsilon$. The readers can convince themselves using the integration by parts in (ii) that the derivative of this function g is the distribution $g' = f'\Upsilon + f(0)\delta$.

Definition 2.2 (Impulse response)
The **impulse response** of a system is its response to the Dirac delta function as input.

Example 2.1
Back to Example 1.1, the impulse response of the mechanical system, i.e. its position, is the **ramp function** $x(t) = t\Upsilon(t)$, since delta function corresponds to speed $v_0 = 1$.

Remark: We define the differential operator $D : \mathcal{F}(\mathbb{R}, \mathbb{R}) \to \mathcal{F}(\mathbb{R}, \mathbb{R}) \quad f \mapsto f'$. If a system L commutes with this operator D, i.e. $L \circ D = D \circ L$, then its impulse response is the derivative of its step response. We will see in the next lecture a class of systems commuting with the differential operator.

Example 2.2

The RC circuit commutes with differentiation. Thus its impulse response is:

$$h(t) = \frac{1}{RC} \exp\left(-\frac{t}{RC}\right) \Upsilon(t)$$